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Creating Symmetry: the Artful Mathematics of Wallpaper Patterns. By Frank A. Farris. Princeton University Press, 2015. Pp. 248. Price GBP 27.95, USD 35.00 (hardcover). ISBN 9780691161730.

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Crystallographers are probably most familiar with tilings, uniformly discrete sets of points, or algebraic systems of vectors and tensors. On occasion, one encounters periodic energy landscapes, but that is not how crystallography is introduced. Frank Farris introduces (two-dimensional) crystallography using functions from the plane to the plane – more precisely, from the complex plane to the complex plane. The text uses these 'wavefunctions' to describe wallpaper symmetries, distorting photographs to get esthetically interesting crystallographic patterns – in a book intended to attract mathematically proficient undergraduate students to crystallography.

Here is how it works.

There is a target photograph, as in Fig. 1. The wavefunction maps the plane to the photograph, and the plane (left of the photograph) is colored so that every point's color is the color of its image point on the photograph. The result is that the domain of the wavefunction has a pattern induced by the wavefunction, and if that function has been set up properly, that pattern is crystallographic.

This example appeared in the discussion of color symmetries. Notice that the purple and green regions are bounded by black curves, which means that all those blackened points are mapped to the black line down the center of the photograph. In all the periodic examples, the entire plane is mapped (repeatedly) into a portion of the photograph.

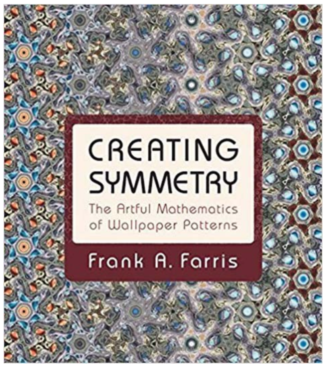
Here is how the patterns are constructed. The complex plane is the set $\mathbb{C} = \{x + iy\colon x, y \in \mathbb{R}\} = \{r\cos\theta + ir\sin\theta\colon r \geq 0 \text{ and } \theta \in [0, 2\pi)\}$, where $i = (-1)^{1/2}$ [so that Euler's formula gives us $\exp(i\theta) = \cos\theta + i\sin\theta$] and $r = (x^2 + y^2)^{1/2}$. Associating a point $(x, y) \in \mathbb{R}^2$ with a complex number $x + iy \in \mathbb{C}$, a function from 2-space to itself – *viz.* from the plane to a photograph – can be represented by a function from $\mathbb{C}$ to $\mathbb{C}$.

Suppose you colored the plane as in Fig. 2 (right image); this is an example of a 'color wheel'. The origin is in the white center, the real axis is horizontal (so that 1 is colored red) while the imaginary axis is vertical (so that i is on the boundary between green and yellow). The polynomial $f(z) = z^2$ maps a point $z = r(\cos\theta + i\sin\theta)$ to a point $z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$, doubling its angle θ while pushing it away from the unit circle. If we colored the plane to show where this wavefunction f sends points, we would get the image on the left of Fig. 2.

Heading towards periodic patterns, if $g(z) = g(x + iy) = \exp(2\pi iy) = \cos(2\pi y) + i\sin(2\pi y)$, we get Fig. 3 (on the left): a point (x, y) is mapped to $\exp(2\pi iy) = \cos(2\pi y) + i\sin(2\pi y)$, so we get horizontal bars. For example, starting from $1 = 1 + 0i$ in the middle of a red bar [as it is mapped to $1 = 1 + 0i$ in Fig. 2 (right)], i is on the next boundary up between yellow and green, as that is where g maps $i = 0 + 1i$.

Farris exhibits a function (which happens to be $h(z) = (1/3)\{\exp(2\pi iy) + \exp[2\pi i(3^{1/2}x - y)/2] + \exp[2\pi i(-(3)^{1/2}x - y)/2]\}$) to obtain a periodic coloring of the domain, the color of a point indicating where in Fig. 2 (right image) the point was sent. Now, $h(0) = 1$ (so that the origin is colored red) while $h(1) = h[1/2 + i(3)^{1/2}/2] = \ldots h[1/2 - i(3)^{1/2}/2] = 0$ (so that 1 is colored white), and repeating for other points we obtain Fig. 3 (right).

The primary construction in this book is to convert symmetries into formulas for complex functions. A *symmetry* of a function $f\colon \mathbb{C} \to \mathbb{C}$ is a function $\alpha\colon \mathbb{C} \to \mathbb{C}$ such that



for every $z \in \mathbb{C}$, $f[\alpha(z)] = f(z)$. For example, the *complex conjugacy function* $x + iy \mapsto x - iy$ (often denoted $\bar{z}$) captures reflection across the real axis, so that $f(\bar{z}) = f(z)$ for all $z \in \mathbb{C}$ if and only if the pattern of f has the horizontal axis as a mirror. Similarly, the rotation by θ degrees about the origin is captured by the function $z \mapsto \exp(i\theta)z$, so that $f[\exp(i\theta)z] = f(z)$ if and only if the pattern of f is preserved by rotation by θ around the origin. For example, Farris chose $a, b \in \mathbb{C}$ (which he did not reveal) to obtain a function

$$j(z) = z^5 + \bar{z}^5 + a(z^6\bar{z} + z\bar{z}^6) + b(z^4\bar{z}^{-6} + \bar{z}^4 z^{-6})$$

which describes how the points of Fig. 4 (left) are mapped to the rhododendron of Fig. 4 (right). One can verify that $j(\bar{z}) = j(z)$ and, with a little more work, that $j[\exp(2\pi i/5)z] = j(z)$.

[We should mention that this is not the usual way complex analysis works. Complex analysis usually considers those functions for which the derivative as a limit

$$f'(z) = \lim_{w \to 0} \frac{f(z + w) - f(z)}{w}$$

is well defined, *i.e.* as the complex variable w tends to 0, the quotient tends to one particular complex number $f'(z)$. Such functions are called *analytic*, and the peculiar properties of analytic functions are what makes complex analysis like a magic act. But most of the periodic functions in the book are not analytic, and thus much of the machinery of complex analysis cannot be applied to them.]

Once the system for having the formulas reflect the symmetries is set up, one can generate all the point groups, frieze groups, wallpaper groups, color groups, and more, by imposing restrictions on the formulas for functions that generate the patterns with those symmetries. One can also demonstrate that the list is exhaustive (at least for dimension 2).

The motivation is largely generating artwork having desired patterns. Several chapters are of the form: here is a class of symmetry groups we are interested in, here are the restrictions on the coefficients that entail these restrictions, here is an example or two – a photograph and the resulting pattern – and notice that the pattern has the desired symmetries. On rare occasions, the text discusses a little about reconstructing the original photo from the pattern – which gives an idea of how the function works – but usually the focus is on the pattern and its symmetries.

There are five special topics. Farris approaches quasi-periodicity in the classical way, by introducing a (two-dimensional) almost periodic wavefunction with (almost) fivefold symmetry. Color symmetries are handled by rotating

Figure 1
The pattern on the left shows where the wavefunction maps points to the photograph on the right.

Figure 2
The polynomial $f(z) = z^2$ maps the plane on the left to the plane on the right, the latter being a standard color wheel. The colors of the points on the left plane – the domain – indicate where those points are sent to on the right plane. Colors occur twice on the left image as f doubles the angle; the white region is proportionately larger as f pushes the points away from the unit circle (alas, the figures are not on the same scale).

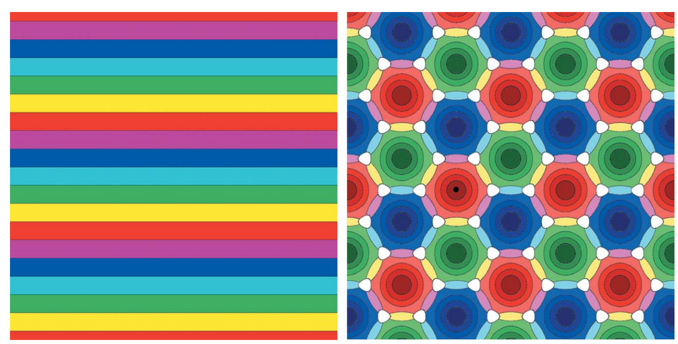


Figure 3
The function g maps the left plane to the color wheel obtaining a pre-image periodic in one direction. The function h maps the right plane to the color wheel, obtaining a pattern periodic in two (and more) directions.

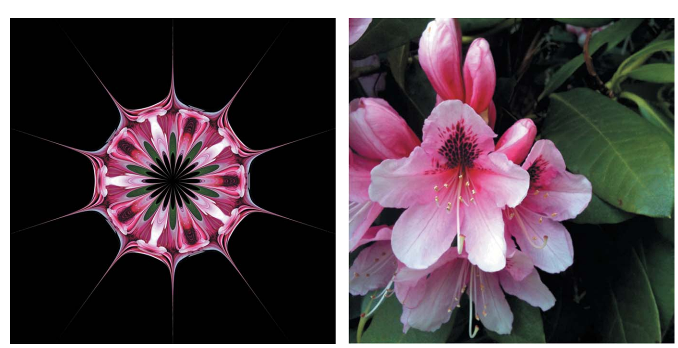


Figure 4
The pattern on the left exhibits both rotational and reflection symmetries.

the color wheel: assuming that there are c colors in the color wheel, a function $f\colon \mathbb{C} \to \mathbb{C}$ has a *color-turning symmetry* at the origin if, for each $z \in \mathbb{C}$, $f[\exp(2\pi i/c)z] = \exp(2\pi i/c)f(z)$: the c-fold rotation $\exp(2\pi i/c)$ left of the equals sign rotating the plane while the c-fold rotation right of the equals sign rotates the color wheel.

Local symmetries are of particular interest. A wavefunction may be the sum of two wavefunctions, $f = g + h$, where h has α as a symmetry at a point $z_0 \in \mathbb{C}$ but g does not, so that α is not a symmetry of f. But if $\lim_{z\to z_0} g(z) = 0$, then near z_0, the symmetry nearly holds, but as one moves away from z_0, the symmetry degrades (see Fig. 5).

A chapter on stereographic projection focuses on displaying the symmetries of polyhedra, and once again, the focus is on the resulting artwork. And for those interested in non-Euclidean geometry, there is a chapter on patterns on Poincaré's hyperbolic half-plane.

Altogether, this is a very interesting and attractive book. The target audience – mathematically capable undergraduates – should find it enticing and accessible. Its textbook structure makes it usable for a seminar course or independent study as well as for private reading; however, it is dense reading. The primary motivation is the artwork, and students may be motivated to obtain some of the software and play with images on their own or in a project. Farris does not assume that the reader is familiar with complex numbers, groups, rings, Fourier series, or even linear algebra: the book is developed in an informal style with some of the results proven, some relegated to exercises (there are 65 exercises), and some to be taken for granted.

Farris writes that he had three audiences in mind: 'the working mathematician, the advanced undergraduate and the brave mathematical adventurer'. For mathematicians, the artistic context is the novelty: the mathematics is familiar, but used in ways even mathematical crystallographers will probably find unfamiliar. For mathematical adventurers – which clearly includes crystallographers, mathematical amateurs, computer graphics engineers and hopefully some artists – while the book does require some calculus, '. . . everything [else] we need will be built slowly as we go'.

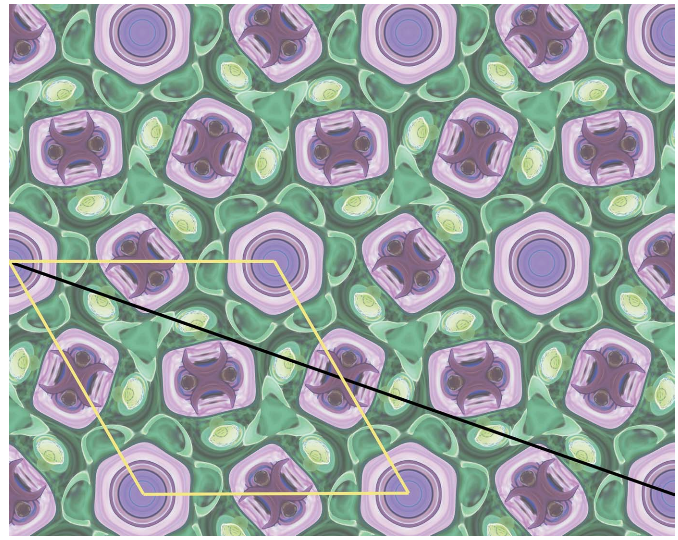


Figure 5
At the intersection of the yellow and black lines, the black line is nearly a mirror for the dark purple oval with the four-legged figure, less so for the surrounding light purple region, and not at all outside.

I do have two laments. In a number of places, Farris does not adequately explain what he is doing, and it takes a lot of reading and reverse engineering to figure out what is going on. In particular, one gaping hole in the book is what the wavefunctions are doing: given a photograph, the pattern is a distortion of the photograph but the wavefunction maps purple points to purple points and mauve points to mauve points. A reader may find it rewarding to track what the wavefunction does to the entire picture, for that should give an idea of what the wavefunction does.

Second, while there are a number of nice references, there aren't citations for anyone interested in pursuing a subject. This can give readers a false impression of the importance of a topic, but it may be harder for a reader to find sources for further study.

But, altogether, this is a fun and informal book that should be interesting to readers at all levels, and certainly every liberal arts institution should have a copy in their library.